

On the equation $n_1 \dots n_{k+1} = n_{k+2} \dots n_{2(k+1)}$ and some variant with restricted unknowns

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ABSTRACT. Let $k \geq 1$ and let $N_k(B)$ denote the number of solutions of the equation

$$n_1 \dots n_{k+1} = n_{k+2} \dots n_{2(k+1)}$$

with unknowns verifying $1 \leq n_i \leq B$, $1 \leq i \leq 2(k+1)$. When $k = 1$, very precise estimates are known. When $k \geq 2$, we prove using elementary arithmetic that

$$B^{k+1}(\log B)^{k^2} \ll_k N_k(B) \ll_k B^{k+1}(\log B)^{k^2+2k-2}.$$

We further study the number of solutions $N(B, F)$ of the equation $n_1 n_2 = n_3 n_4$, where $1 \leq n_1 \leq B$, $1 \leq n_3 \leq B$, $n_2, n_4 \in F$ and $F \subset [1, B]$ is a factor closed set. Let $F = \{m = p_1^{\varepsilon_1} \dots p_k^{\varepsilon_k}, \varepsilon_j \in \{0, 1\}, 1 \leq j \leq k\}$, where $p_1 < \dots < p_k$ are prime numbers, and let $B \geq p_1 p_2 \dots p_k$. We prove that

$$2^k B \left(1 + 2 \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{2^{\omega(n)} n}\right) \leq N(B, F) \leq B \left(\frac{5}{2}\right)^k \left(1 + C \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{5^{\omega(n)} n}\right) + 4^k,$$

where C is a numerical constant. Our approach is much related with the multiplicative sums $\sum_{j \leq x} d_k(jn) d_k(jm)$, $(n, m) = 1$, where d_k is the Piltz divisor function.

1. Introduction-Results.

Let $k \geq 1$ and consider the equation

$$(1.1) \quad n_1 \dots n_{k+1} = n_{k+2} \dots n_{2(k+1)}.$$

Let $N_k(B)$ denote the number of solutions of equation (1.1) with unknowns verifying $1 \leq n_i \leq B$, $1 \leq i \leq 2(k+1)$.

When $k = 1$, we have the following very precise estimate improving upon previous similar estimate by Ayyad, Cochrane and Zheng [1],

$$(1.2) \quad N_1(B) = \frac{12}{\pi^2} B^2 \log B + C B^2 + \mathcal{O}(B^{\frac{547}{416} + \varepsilon}),$$

where

$$C = \left(\frac{36}{\pi^2} \gamma_0 - 2\gamma_1 + \frac{12}{\pi^2} \gamma_2 - \frac{24}{\pi^2} - 1\right) = 0.511317447\dots$$

and $\gamma_1 = \frac{36}{\pi^2} \zeta'(2)$, $\gamma_2 = \frac{3}{2} - \gamma_0 - \frac{\pi^2}{12}$, γ_0 being Euler's constant. See Shi [15, Theorem 1]. The proof connects the evaluation of $N_1(B)$ with the error term in the Dirichlet divisor problem, and only involves elementary arithmetic. Some more simplification were provided by Liu and Zhai [10].

When $k > 1$, the problem of estimating $N_k(B)$ becomes much more complicated and such a precise estimate seems out of reach. In an unpublished work, Granville and Soundarajan [4] proved the following estimate

$$(1.3) \quad N_k(B) \sim c(k) B^{k+1} (\log B)^{k^2},$$

where the constant $c(k)$ depends on k only. Very recently, Harper, Nikeghbali and Radziwill, and independently Heap and Lindqvist, recovered Granville and Soundarajan's estimate as a consequence of estimates of moments of random multiplicative functions they obtained in [5] and

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[7] respectively. This last question was investigated by Granville and Soundarajan in the early years 2000' and by Conrey and Gamburd [2] notably, and attracts many interest because of some (heuristic and numerical) analogy between random matrix theory and Riemann zeta-function. These objects are also interesting from a purely probabilistic point of view, and deserve more investigation. Their proof use contour integral representation for the moments, and an integral trick, see Ingham [8, Th. G], see also Conrey and Gamburd [2, (42)], but does not seem to provide estimates of the remainder term as given in (1.2) for $k = 1$.

In this work, our aim is to show that the arithmetical approach performed in [15] can be developed to estimate $N_k(B)$, and next, that it can also be applied to the study of some variant of the initial equation, namely with restrictions imposed on the range of the unknowns. The underlying motivation is to develop this approach to the study of similar equations and to extend the accurate estimate (1.2) to the case of a larger number of unknowns.

It turns out that by using this alternative method we could, a bit unexpectedly, recover almost all from estimate (1.3). In particular we propose a rather simple and short proof of the lower bound part. More precisely, by means of simple arithmetical devices we show

Proposition 1.1. *We have for $k > 2$,*

$$B^{k+1}(\log B)^{k^2} \ll_k N_k(B) \ll_k B^{k+1}(\log B)^{k^2+2k-2}.$$

Here and in what follows the notation $x \ll y$ (resp. $x \ll_k y$) means that $x \leq Cy$ (resp. $x \leq C_k y$) where C is an absolute constant (resp. C_k a constant depending on k only), and which may change of value at each occurrence. Only for the upper bound, there is a loss in the exponent ($k^2 + 2k - 2$ instead of k^2). We believe this gap can be filled by refining our approach.

Although we did not pay attention to the constants involved, we shall mention that in all used estimates, the constants are known explicitly. The proof directly relies on estimates on a kind of multiplicative correlation of the divisor function, namely the sums

$$(1.4) \quad \sum_{j \leq x} d(jn)d(jm), \quad (n, m) = 1,$$

and their analog for the Piltz divisor function d_k . This is one interesting aspect of this approach, and we expect to somewhat motivate their study. We were unable to rely the sums (1.4) to some work in the existing literature.

Our starting point for $k = 2$ is indeed ((2.5), (2.6)) the following formulation of $N_2(B)$,

$$N_2(B) = B\#(\mathcal{M}(B)) + 2 \sum_{\substack{1 \leq n < m \leq B \\ (n, m) = 1}} \left[\frac{B}{m} \right] \left(\sum_{j=1}^{[B^2/m]} d(jn)d(jm) \right).$$

Here $\mathcal{M}(B)$ is the set of integers m of type $m = m_1.m_2$ with $1 \leq m_1 \leq B$, $1 \leq m_2 \leq B$, and Ford's recent estimates of $\#(\mathcal{M}(B))$ are needed (see Lemma 2.2).

A brief analysis of the problem can be presented as follows. By the Busche-Ramanujan formula [13, p. 25],

$$(1.5) \quad d(k)d(l) = \sum_{\tau | (k, l)} d\left(\frac{kl}{\tau^2}\right),$$

we can put (1.4) under the more concise form

$$(1.6) \quad \sum_{j \leq x} d(jn)d(jm) = \sum_{u \leq x} d(u^2 mn) \left[\frac{x}{u} \right],$$

The problem thus reduces to the study of divisors of a reducible quadratic polynomial, more precisely to the study of the sum $\sum_{v \leq Y} d(v^2 a)$ where a is any positive integer. Except for the value $a = 1$, where we know that (see [9, (14.29)] for instance)

$$(1.7) \quad \sum_{v \leq y} d(v^2) = A_1 y \log^2 y + \mathcal{O}(y \log y), \quad y > 1,$$

we could not find any suitable reference. In place we proceeded with directly estimating the sum (1.4). In doing so, we could work out short and simple proofs. The study of the summatory function of $d(v^2a)$ and sums (1.4) is of independent interest and will be made elsewhere (see in particular Section 6).

We also study the following variant of the initial equation. Let F be a finite set of integers. We assume that F is factor closed ($d|k \Rightarrow d \in F$ for all $k \in F$). See Haukkanen, Wang and Sillanpää [6] and references therein for this notion and extensions, as well as Weber [17]. Let B be such that $F \subset [1, B]$.

Let $N(B, F)$ denote the number of integers solutions of the restricted equation

$$(1.8) \quad n_1 n_2 = n_3 n_4,$$

where the unknowns verify $1 \leq n_1 \leq B$, $1 \leq n_3 \leq B$, $n_2, n_4 \in F$.

Proposition 1.2. *We have*

$$N(B, F) \leq 2B^2 \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{n} + B^2.$$

See also Proposition 6.4 where another estimate, more depending on F , is obtained. We deduce

Theorem 1.3. *Suppose that $F = \{m_1 m_2 : 1 \leq m_1, m_2 \leq [B^\alpha]\}$ with $\alpha \in [0, 1/2]$. We have*

$$N(B, F) \leq \left(1 + \frac{12}{\pi^2} + 2 \sum_{\substack{B^\alpha < m \leq B^{2\alpha} \\ m \in F}} \frac{1}{m}\right) B^{1+2\alpha}.$$

Let us consider the following typical example of FC set. Let

$$F = \left\{m = p_1^{\varepsilon_1} \dots p_k^{\varepsilon_k}, \varepsilon_j \in \{0, 1\}, 1 \leq j \leq k\right\},$$

where $p_1 < \dots < p_k$ are prime numbers, and let $B \geq p_1 p_2 \dots p_k$.

Theorem 1.4. *We have*

$$2^k B \left(1 + 2 \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{2^{\omega(n)} n}\right) \leq N(B, F) \leq B \left(\frac{5}{2}\right)^k \left(1 + C \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{5^{\omega(n)} n}\right) + 4^k,$$

where C is a numerical constant. Assume that $\sum_{j=1}^k \frac{1}{p_j} \leq \frac{5}{2}$. Then

$$2^k B \left(1 + \sum_{j=1}^k \frac{1}{p_j}\right) \leq N(B, F) \leq B \left(\frac{5}{2}\right)^k \left(1 + \frac{2C}{5} \left(\sum_{j=1}^k \frac{1}{p_j}\right)\right) + 4^k.$$

And for large k ,

$$N(B, F) \ll B \left(\frac{5}{2}\right)^k \left(1 + \sum_{j=1}^k \frac{1}{p_j}\right).$$

The paper is organized as follows. In the next Section, some preparatory Lemmas are established, next in order to describe our method, we show how it applies to treat the case $k = 2$. In Section 3 we consider the general case and indicate which modifications are necessary. In Sections 4 and 5 we prove Theorems 1.3 and 1.4 respectively. Finally in Section 6, some complements are given.

2. Preliminary Lemmas.

Let F be a factor closed set such that $F \subset [1, B]$, and recall that $N(B, F)$ denotes the number of integers solutions of the restricted equation (1.8).

Lemma 2.1. *We have*

$$N(B, F) = \sum_{\substack{(n, m)=1 \\ n, m \in F}} \left(\sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1 \right) \left[\frac{B}{n \vee m} \right].$$

In particular

$$N(B, F) \leq 2B^2 \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{n} + B^2.$$

The above bound fits with the basic but important case $F = [1, B]$.

Proof. Equation (1.8) means that $n_1 n = n_3 m$, where $n_2 = dn$, $n_4 = dm$, $d = (n_2, n_4)$ and $(n, m) = 1$. Note that since F is factor closed, we have $n, m \in F$.

Now given $n, m \in F$ fixed and such that $(n, m) = 1$, the number of integers $n_2, n_4 \in F$, which write $n_2 = dn$, $n_4 = dm$ for some $d \geq 1$, is

$$\sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1.$$

Further, the solutions of $(n_1, n_3) = (n, m)$ in the unknowns $1 \leq n_1 \leq B$, $1 \leq n_3 \leq B$ are trivially $(n_1, n_3) = (\lambda n, \lambda m)$, with $1 \leq \lambda \leq \left\lfloor \frac{B}{n \vee m} \right\rfloor$.

Hence the number of solutions in the unknowns $1 \leq n_1 \leq B$, $1 \leq n_3 \leq B$, $n_2, n_4 \in F$ and verifying $\frac{n_1}{n_3} = \frac{n_4}{n_2} = \frac{m}{n}$, is

$$(2.1) \quad \left(\sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1 \right) \left[\frac{B}{n \vee m} \right].$$

Note that when $F = [1, B]$, this simplifies and one gets $\left\lfloor \frac{B}{n \vee m} \right\rfloor^2$, which for $n = m = 1$ reduces to B^2 .

Also (see [15, (4)])

$$N_1(B) = \sum_{\substack{(n, m)=1 \\ 1 \leq n, m \leq B}} \left[\frac{B}{n \vee m} \right]^2 = B^2 + 2 \sum_{\substack{(n, m)=1 \\ 1 \leq n < m \leq B}} \left[\frac{B}{m} \right]^2.$$

In our case we have something slightly more complicated, namely

$$(2.2) \quad N(B, F) = \sum_{\substack{(n, m)=1 \\ n, m \in F}} \left(\sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1 \right) \left[\frac{B}{n \vee m} \right].$$

As $F \subset [1, B]$, we have the obvious bound

$$\sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1 \leq \left[\frac{B}{n \vee m} \right].$$

And so,

$$N(B, F) \leq \sum_{\substack{(n, m)=1 \\ n, m \in F}} \left[\frac{B}{n \vee m} \right]^2 = B^2 + 2 \sum_{\substack{(n, m)=1 \\ n < m \\ n, m \in F}} \left[\frac{B}{m} \right]^2.$$

Plainly,

$$\mathcal{B} := \sum_{\substack{(n, m)=1 \\ n, m \in F}} \left[\frac{B}{m} \right]^2 \leq B^2 \sum_{m \in F} \frac{1}{m^2} \sum_{\substack{(n, m)=1 \\ n < m}} 1 = B^2 \sum_{\substack{m \in F \\ m \neq 1}} \frac{\phi(m)}{m^2} \leq B^2 \sum_{\substack{m \in F \\ m \neq 1}} \frac{1}{m}.$$

Here we used the elementary inequality, $\phi(n) \leq n$. Hence the claimed bound. \square

Lemma 2.2. ([3]) *We have*

$$\#(\mathcal{M}(B)) \asymp \frac{B^2}{(\log B)^\delta (\log \log B)^{3/2}},$$

where

$$\delta = 1 - \frac{1 + \log \log 2}{\log 2}.$$

Further,

$$N_1(B) = \sum_{m \in \mathcal{M}(B)} d_2^2(m).$$

Consider first the case $k = 2$, namely the equation

$$(2.3) \quad n_1 n_2 n_3 = n_4 n_5 n_6$$

with unknowns verifying $1 \leq n_i \leq B$, $i = 1, 2, 3, 4, 5, 6$. Let $N_2(B)$ denote the number of solutions.

Lemma 2.3. *We have*

$$N_2(B) = \sum_{\substack{(n,m)=1 \\ 1 \leq n, m \leq B}} \left(\sum_{\substack{n_2, n_4 \in \mathcal{M}(B) \\ \frac{n_4}{n_2} = \frac{n}{m}}} 1 \right) \left[\frac{B}{n \vee m} \right].$$

Proof. Equation (2.3) can be rewritten under the form $n_1 n_2 = n_3 n_4$, with $1 \leq n_i \leq B$, $i = 1, 3$, $n_2, n_4 \in \mathcal{M}(B)$. Hence (2.3) means

$$n n_2 = m n_4$$

where this time, we write $n_1 = dn$, $n_3 = dm$, with $d = (n_1, n_3)$ and $(n, m) = 1$, and $F = \mathcal{M}(B)$. Note that $\mathcal{M}(B)$ is a FC set. Hence

$$\frac{n_1}{n_3} = \frac{n_4}{n_2} = \frac{n}{m}.$$

The number of solutions in the unknowns $1 \leq n_1 \leq B$, $1 \leq n_3 \leq B$, $n_2, n_4 \in \mathcal{M}(B)$ and verifying $\frac{n_1}{n_3} = \frac{n_4}{n_2} = \frac{n}{m}$, is

$$\left[\frac{B}{n \vee m} \right] \left(\sum_{\substack{n_2, n_4 \in \mathcal{M}(B) \\ \frac{n_4}{n_2} = \frac{n}{m}}} 1 \right).$$

Hence,

$$N_2(B) = \sum_{\substack{(n,m)=1 \\ 1 \leq n, m \leq B}} \left[\frac{B}{n \vee m} \right] \left(\sum_{\substack{n_2, n_4 \in \mathcal{M}(B) \\ \frac{n_4}{n_2} = \frac{n}{m}}} 1 \right).$$

□

We shall now first show that

$$(2.4) \quad B^3 \log^4 B \ll N_2(B) \ll B^3 \log^6 B.$$

Proof of (2.4). Lemma 2.3 implies that

$$(2.5) \quad N_2(B) = B \#(\mathcal{M}(B)) + 2 \sum_{\substack{n, m, (n,m)=1 \\ 1 \leq n < m \leq B}} \left[\frac{B}{m} \right] \left(\sum_{\substack{n_2, n_4 \in \mathcal{M}(B) \\ \frac{n_4}{n_2} = \frac{n}{m}}} 1 \right) := B \#(\mathcal{M}(B)) + 2\mathcal{B}_1.$$

Let us consider the inner sum, we get

$$(2.6) \quad \sum_{\substack{\frac{n_4}{n_2} = \frac{n}{m} \\ n_2, n_4 \in \mathcal{M}(B)}} 1 = \sum_{\substack{mn'_1 n'_2 = n n'_3 n'_4 \\ n'_i \in [1, B]}} 1 = \sum_{\substack{c = nm \\ nm | c}}^{nB^2} \left(\sum_{mn'_1 n'_2 = c} 1 \right) \left(\sum_{nn'_3 n'_4 = c} 1 \right)$$

$$(2.7) \quad = \sum_{j=1}^{[B^2/m]} d(jn) d(jm).$$

Upper bound. We have

$$\sum_{\substack{\frac{n_4}{n_2} = \frac{n}{m} \\ n_2, n_4 \in \mathcal{M}(B)}} 1 \leq d(n)d(m) \sum_{j=1}^{[B^2/m]} d(j)^2 \ll \frac{B^2}{m} (\log B)^3 d(n)d(m),$$

where we have used $d(nm) \leq d(n)d(m)$ and Ramanujan's well known estimate $\sum_{n \leq x} d^2(n) \sim Cx(\log x)^3$. See [14, (3)], [20, (1.70)]. Hence we deduce that

$$\begin{aligned} \mathcal{B}_1 &\ll (B \log B)^3 \sum_{2 \leq m \leq B} \sum_{\substack{n < m \\ (n, m) = 1}} \frac{d(n)d(m)}{m^2} \leq (B \log B)^3 \sum_{2 \leq m \leq B} \frac{d(m)}{m^2} \sum_{n < m} d(n) \\ (2.8) \quad &\leq B^3 (\log B)^4 \sum_{2 \leq m \leq B} \frac{d(m)}{m} \leq B^3 (\log B)^6, \end{aligned}$$

by using the estimate $\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2} \log^2 x + O(\log x)$. From (2.5)–(2.8) and Lemma 2.2 we get the upper bound

$$N_2(B) \ll B^3 \log^6 B.$$

Lower bound. We have

$$\sum_{\substack{\frac{n_4}{n_2} = \frac{n}{m} \\ n_2, n_4 \in \mathcal{M}(B)}} 1 \geq \sum_{j=1}^{[B^2/m]} d(j)^2 \gg \frac{B^2}{m} \log^3 B,$$

where we have used sub-multiplicativity of d_2 and the same estimate as before. Then

$$\mathcal{B}_2 \gg \sum_{2 \leq m \leq B} \sum_{\substack{n < m \\ (n, m) = 1}} \left[\frac{B}{m} \right] \frac{B^2}{m} \log^3 B \gg B^3 \log^3 B \sum_{2 \leq m \leq B} \frac{1}{m^2} \sum_{\substack{n < m \\ (n, m) = 1}} 1 = B^3 \log^3 B \sum_{2 \leq m \leq B} \frac{\phi(m)}{m^2}.$$

Before going further, note that by Abel summation, for $x \geq 2$,

$$\begin{aligned} \sum_{m \leq x} \frac{\phi(m)}{m^2} &= \int_1^x \frac{1}{t} d\left(\sum_{m \leq t} \frac{\phi(m)}{m}\right) = \frac{1}{x} \left(\frac{6}{\pi^2} x + \mathcal{O}(\log x) \right) + \int_1^x \frac{1}{t^2} \left(\frac{6}{\pi^2} t + \mathcal{O}(\log t) \right) dt \\ (2.9) \quad &= \frac{6}{\pi^2} \log x + \mathcal{O}(1), \end{aligned}$$

where the inequality $\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + \mathcal{O}(\log x)$ (see [13], p. 287) is used. By combining,

$$(2.10) \quad \mathcal{B}_2 \gg B^3 (\log^4 B).$$

Hence the lower bound follows immediately from (2.5), (2.10) and Lemma 2.2. \square

3. Proof of Proposition 1.1.

Let $k > 2$ and consider the equation

$$n_1 \dots n_{k+1} = n_{k+2} \dots n_{2(k+1)}$$

with unknowns verifying $1 \leq n_i \leq B$, $1 \leq i \leq 2(k+1)$. Let $N_k(B)$ denote the number of solutions.

Let $\mathcal{M}_k(B)$ be the set of integers m admitting a representation as a product of k factors $m = m_1 \dots m_k$ with $1 \leq m_i \leq B$, $1 \leq i \leq k$. Note that $\mathcal{M}_k(B)$ is again a FC set. Let also $d_k(n)$ denote the Piltz divisor function counting the number of ways to write n as a product of k factors. Note that $d_2(m) = d(m)$. Recall that $v_p(n)$ is the p -valuation of n , $v_p(1) \equiv 0$ and ([13], p. 40)

$$d_k(n) = \prod_p C_{v_p(n)+k-1}^{v_p(n)} = \prod_p \prod_{j=1}^{k-1} \binom{v_p(n)+j}{j}.$$

It follows that d_k is sub-multiplicative.

The lemma below is proved similarly to Lemma 2.3.

Lemma 3.1. *We have*

$$N_k(B) = \sum_{\substack{(n,m)=1 \\ 1 \leq n, m \leq B}} \left[\frac{B}{n \vee m} \right] \left(\sum_{\substack{m_1, m_2 \in \mathcal{M}_k(B) \\ \frac{m_2}{m_1} = \frac{n}{m}}} 1 \right).$$

Upper bound of $N_k(B)$. We have

$$N_k(B) = B \#(\mathcal{M}_k(B)) + 2 \sum_{\substack{(n,m)=1 \\ 1 \leq n < m \leq B}} \left[\frac{B}{m} \right] \left(\sum_{\substack{m_1, m_2 \in \mathcal{M}_k(B) \\ \frac{m_2}{m_1} = \frac{n}{m}}} 1 \right) := B \#(\mathcal{M}_k(B)) + 2\mathcal{B}_k.$$

As for the case $k = 2$,

$$\begin{aligned} \sum_{\substack{m_1, m_2 \in \mathcal{M}_k(B) \\ \frac{m_2}{m_1} = \frac{n}{m}}} 1 &= \sum_{\substack{nm_1, 1 \dots m_{1,k} = nm_2, 1 \dots m_{2,k} \\ m_{1,i}, m_{2,j} \in [1, B]}} 1 = \sum_{\substack{c=n \\ nm|c}}^{nB^k} \left(\sum_{nm_1, 1 \dots m_{1,k} = c} 1 \right) \left(\sum_{mm_2, 1 \dots m_{2,k} = c} 1 \right) \\ &= \sum_{j=1}^{[B^k/m]} d_k(jn) d_k(jm) \leq d_k(n) d_k(m) \sum_{j=1}^{[B^k/m]} d_k(j)^2 \\ (3.1) \quad &\ll_k \frac{B^k}{m} (\log B)^{k^2-1} d_k(n) d_k(m), \end{aligned}$$

where we have used sub-multiplicativity of d_k and the estimate $\sum_{m \leq x} d_k^2(m) = (C_k + o(1)) x \log^{k^2-1} x$. See [9, (9.33)] for instance.

$$(3.2) \quad \mathcal{B}_k \ll_k B^{k+1} (\log B)^{k^2-1} \sum_{2 \leq m \leq B} \sum_{\substack{n < m \\ (n,m)=1}} \frac{d_k(n) d_k(m)}{m^2}.$$

Now, plainly

$$\begin{aligned} \sum_{2 \leq m \leq B} \sum_{\substack{n < m \\ (n,m)=1}} \frac{d_k(n) d_k(m)}{m^2} &\leq \sum_{2 \leq m \leq B} \frac{d_k(m)}{m^2} \sum_{n < m} d_k(n) \ll_k (\log B)^{k-1} \sum_{2 \leq m \leq B} \frac{d_k(m)}{m} \\ &\ll_k (\log B)^{2k-1}, \end{aligned}$$

where we used the fact that $\sum_{m \leq x} d_k(m) \sim C_k x (\log x)^{k-1}$, (see notably Theorem 14.9 in [9]), and further that $\sum_{n \leq x} \frac{d_k(n)}{n} \sim C_k (\log x)^k$. This along with (3.2) implies

$$(3.3) \quad \mathcal{B}_k \ll_k B^{k+1} (\log B)^{k^2+2k-2}.$$

Now $\#(\mathcal{M}_k(B)) \leq B^{k-2} \#(\mathcal{M}(B))$ and by Lemma 2.2, $\#(\mathcal{M}_k(B)) \leq \frac{B^k}{(\log B)^\delta (\log \log B)^{3/2}}$ where $\delta = 1 - \frac{1+\log \log 2}{\log 2}$. By combining,

$$N_k(B) \ll_k B^{k+1} (\log B)^{k^2+2k-2}.$$

Lower bound of $N_k(B)$. Similarly to the proof of the lower bound of $N_2(B)$,

$$\sum_{\substack{m_1, m_2 \in \mathcal{M}_k(B) \\ \frac{m_2}{m_1} = \frac{n}{m}}} 1 = \sum_{j=1}^{[B^k/m]} d_k(jn) d_k(jm) \geq \sum_{j=1}^{[B^k/m]} d_k(j)^2 \gg_k \frac{B^k}{m} (\log B)^{k^2-1}.$$

Hence

$$\mathcal{B}_k \gg_k \sum_{2 \leq m \leq B} \sum_{\substack{n < m \\ (n,m)=1}} \frac{B^{k+1} (\log B)^{k^2-1}}{m^2} = B^{k+1} (\log B)^{k^2-1} \sum_{2 \leq m \leq B} \frac{\phi(m)}{m^2} \gg_k B^{k+1} (\log B)^{k^2},$$

where estimate (2.9) is used again. We note that $\#(\mathcal{M}_k(B)) \leq \frac{B^k}{(\log B)^\delta (\log \log B)^{3/2}}$. By combining,

$$N_k(B) \gg_k B^{k+1} (\log B)^{k^2}.$$

4. Proof of Theorem 1.3.

By assumption, F is a FC set with $F \subset [1, B]$. By (2.2), we have

$$N(B, F) = \#(F) B + 2 \sum_{\substack{(n, m)=1, n < m \\ n, m \in F}} \left[\frac{B}{m} \right] \sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1 := (\#F)[B] + 2\mathcal{B}_0.$$

We get

$$\mathcal{B}_0 = \left(\sum_{\substack{1 < m \leq B^\alpha \\ m \in F}} + \sum_{\substack{B^\alpha < m \leq B^{2\alpha} \\ m \in F}} \right) \left[\frac{B}{m} \right] \sum_{\substack{(n, m)=1 \\ n < m, n \in F}} \sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1 := \mathcal{B}_{01} + \mathcal{B}_{02}.$$

Let us consider the above first sum \mathcal{B}_{01} .

$$\begin{aligned} \mathcal{B}_{01} &= \sum_{1 < m \leq B^\alpha} \left[\frac{B}{m} \right] \sum_{(n, m)=1, n < m} [B^\alpha] = [B^\alpha] \sum_{1 < m \leq B^\alpha} \left[\frac{B}{m} \right] \phi(m) \\ &= B^{1+\alpha} \sum_{m \leq B^\alpha} \frac{\phi(m)}{m} - [B][B^\alpha] + \mathcal{O}\left(B \sum_{2 < m \leq B^\alpha} \frac{\phi(m)}{m}\right) + \mathcal{O}\left(B^\alpha \sum_{2 < m \leq B^\alpha} \phi(m)\right) \\ &= B^{1+\alpha} \left(\frac{6}{\pi^2} B^\alpha + \mathcal{O}(\log B) \right) + \mathcal{O}\left(B^{1+\alpha}\right) \\ &= \frac{6}{\pi^2} B^{1+2\alpha} + \mathcal{O}\left(B^{1+\alpha} \log B\right), \end{aligned}$$

where we used the estimates $\phi(n) \leq n$ and $\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + \mathcal{O}(\log x)$.

Now let us estimate \mathcal{B}_{02} .

$$\begin{aligned} \mathcal{B}_{02} &\leq \sum_{\substack{B^\alpha < m \leq B^{2\alpha} \\ m \in F}} \left[\frac{B}{m} \right] \frac{B^{2\alpha}}{m} \sum_{\substack{n < m, (n, m)=1, \\ n \in F}} 1 \leq B^{1+2\alpha} \sum_{\substack{B^\alpha < m \leq B^{2\alpha} \\ m \in F}} \frac{\phi(m)}{m^2} \\ &\leq B^{1+2\alpha} \sum_{\substack{B^\alpha < m \leq B^{2\alpha} \\ m \in F}} \frac{1}{m}. \end{aligned}$$

By reporting, we have

$$N(B, F) \leq \left(1 + \frac{12}{\pi^2} + 2 \sum_{\substack{B^\alpha < m \leq B^{2\alpha} \\ m \in F}} \frac{1}{m} \right) B^{1+2\alpha}.$$

5. Proof of Theorem 1.4

Let $m \in F$. Given an integer $a \geq 2$, we note $\langle a \rangle = \{p : p|a\}$. Recall that $\omega(n)$ denotes the prime divisor function, and $\omega(1) = 0$. Consider first the sum

$$\sigma(n, m) = \sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1.$$

Then $\frac{n_4}{n_2} = \frac{m}{n}$ gives rise to $n_4 = \lambda m$, $n_2 = \lambda n$. As $\lambda, \lambda m, \lambda n \in F$, it follows by definition of F that $\langle \lambda \rangle \cap \langle m \rangle = \langle \lambda \rangle \cap \langle n \rangle = \emptyset$. Thus $\langle \lambda \rangle \subset \{p_1, \dots, p_k\} - \langle m \rangle - \langle n \rangle$. And so we have

$$\sigma(m, n) = 2^{k-\omega(m)-\omega(n)}$$

for all $n, m \in F$. By Lemma 2.1, since $F \subset [1, B]$,

$$\begin{aligned} N(B, F) &= \sum_{\substack{(n, m)=1 \\ n, m \in F}} \left(\sum_{\substack{n_2, n_4 \in F \\ \frac{n_4}{n_2} = \frac{m}{n}}} 1 \right) \left[\frac{B}{n \vee m} \right] = \sum_{\substack{(n, m)=1 \\ n, m \in F}} 2^{k-\omega(m)-\omega(n)} \left[\frac{B}{n \vee m} \right] \\ (5.1) \quad &= BY + \mathcal{O}\left(\sum_{\substack{(n, m)=1 \\ n, m \in F}} 2^{k-\omega(m)-\omega(n)} \right), \end{aligned}$$

where we set

$$Y := \sum_{\substack{(n,m)=1 \\ n,m \in F}} 2^{k-\omega(m)-\omega(n)} \frac{1}{n \vee m}.$$

Moreover,

$$N(B, F) \geq 2^k B + 2B \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{n} \sigma(n, 1) = 2^k B + 2^{k+1} B \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{2^{\omega(n)} n}.$$

That is, the lower bound is obtained. Next consider the upper bound for $N(B, F)$. Note that $\frac{1}{n \vee m} = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{m}} \left(\frac{n \wedge m}{n \vee m} \right)^{1/2}$. From the relation $e^{-|\vartheta|} = \int_{\mathbb{R}} e^{i\vartheta t} \frac{dt}{\pi(t^2+1)}$, it follows that

$$(5.2) \quad \left(\frac{n}{m} \right)^s = \int_{\mathbb{R}} \frac{1}{n^{-ist} m^{ist}} \frac{dt}{\pi(t^2+1)} \quad (m \geq n).$$

Take $s = 1/2$. We get

$$\frac{1}{n \vee m} = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{m}} \int_{\mathbb{R}} \frac{1}{n^{-it/2} m^{it/2}} \frac{dt}{\pi(t^2+1)}.$$

Recall that μ denotes the Möbius function and that

$$(5.3) \quad \sum_{d|n} \mu(d) = \delta(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

By reporting

$$\begin{aligned} Y &= 2^k \sum_{\substack{(n,m)=1 \\ n,m \in F}} \frac{1}{2^{\omega(n)} \sqrt{n}} \frac{1}{2^{\omega(m)} \sqrt{m}} \int_{\mathbb{R}} \frac{1}{n^{-it/2} m^{it/2}} \frac{dt}{\pi(t^2+1)} \\ &= 2^k \sum_{d \in F} \mu(d) \sum_{\substack{n,m \in F \\ d|n, d|m}} \frac{1}{2^{\omega(n)} \sqrt{n}} \frac{1}{2^{\omega(m)} \sqrt{m}} \int_{\mathbb{R}} \frac{1}{n^{-it/2} m^{it/2}} \frac{dt}{\pi(t^2+1)} \\ &= 2^k \sum_{d \in F} \mu(d) \int_{\mathbb{R}} \left| \sum_{\substack{n \in F \\ d|n}} \frac{1}{2^{\omega(n)} \sqrt{n}} \frac{1}{n^{it/2}} \right|^2 \frac{dt}{\pi(t^2+1)}. \end{aligned}$$

If $n \in F$ and $d|n$, then $n = \nu d$ and $\langle \nu \rangle \subset \{p_1, \dots, p_k\} - \langle d \rangle$. Thus

$$\sum_{\substack{n \in F \\ d|n}} \frac{1}{2^{\omega(n)} \sqrt{n}} \frac{1}{n^{it/2}} = \frac{1}{2^{\omega(d)} d^{(1+it)/2}} \sum_{\substack{\nu \in F \\ (\nu, d)=1}} \frac{1}{2^{\omega(\nu)} \sqrt{\nu}} \frac{1}{\nu^{it/2}}.$$

By Lemma 2.4 in [18],

$$\int_{\mathbb{R}} \left| \sum_{n=1}^N x_n n^{ist} \right|^2 \frac{dt}{\pi(t^2+1)} = \sum_{j=1}^N (j^{2s} - (j-1)^{2s}) \left| \sum_{\mu=j}^N \frac{x_\mu}{\mu^s} \right|^2$$

for any real $s \geq 0$. By Lemma 2.5 in [18],

$$\sum_{j=1}^N (j^{2s} - (j-1)^{2s}) \left| \sum_{\mu=j}^N \frac{x_\mu}{\mu^s} \right|^2 \leq \begin{cases} C_s \sum_{\mu=1}^N |x_\mu|^2 \mu^{3/2-2s} & \text{if } 0 < s < 1/4, \\ C \sum_{\mu=1}^N |x_\mu|^2 \mu \log \mu & \text{if } s = 1/4, \\ C_s \sum_{\mu=1}^N |x_\mu|^2 \mu & \text{if } s > 1/4, \end{cases}$$

for any $s > 0$ and complex numbers x_j , $j = 1, \dots, N$.

Therefore,

– If $d = 1$, we have

$$\int_{\mathbb{R}} \left| \sum_{n \in F} \frac{1}{2^{\omega(n)} \sqrt{n}} \frac{1}{n^{it/2}} \right|^2 \frac{dt}{\pi(t^2+1)} \leq C \sum_{n \in F} \frac{1}{2^{2\omega(n)} n} n = \sum_{n \in F} \frac{1}{4^{\omega(n)}} = \left(\frac{5}{4} \right)^k,$$

since

$$\sum_{\nu \in F} \frac{1}{4^{\omega(\nu)}} = \sum_{y=0}^k \sum_{\nu \in F, \omega(\nu)=y} 4^{-y} = \sum_{y=0}^k C_k^y 4^{-y} = \left(\frac{5}{4}\right)^k.$$

– If $d > 1$, $d \in F$, then using strong additivity of function ω ,

$$\begin{aligned} \int_{\mathbb{R}} \left| \sum_{\substack{n \in F \\ d|n}} \frac{1}{2^{\omega(n)} \sqrt{n}} \frac{1}{n^{it/2}} \right|^2 \frac{dt}{\pi(t^2 + 1)} &= \frac{1}{2^{2\omega(d)} d} \int_{\mathbb{R}} \left| \sum_{\substack{\nu \in F \\ (\nu, d)=1}} \frac{1}{2^{\omega(\nu)} \sqrt{\nu}} \frac{1}{\nu^{it/2}} \right|^2 \frac{dt}{\pi(t^2 + 1)} \\ &\leq \frac{C}{2^{2\omega(d)} d} \sum_{\substack{\nu \in F \\ (\nu, d)=1}} \frac{1}{2^{2\omega(\nu)} \nu} \nu = \frac{C}{2^{2\omega(d)} d} \sum_{\substack{\nu \in F \\ (\nu, d)=1}} \frac{1}{2^{2\omega(\nu)}}. \end{aligned}$$

And if $d > 1$, $d \in F$, then

$$\sum_{\substack{\nu \in F \\ (\nu, d)=1}} \frac{1}{2^{2\omega(\nu)}} = \sum_{y=0}^{k-\omega(d)} \sum_{\nu \in F, \omega(\nu)=y} 4^{-y} = \sum_{y=0}^{k-\omega(d)} C_{k-\omega(d)}^y 4^{-y} = \left(\frac{5}{4}\right)^{k-\omega(d)} = \left(\frac{5}{4}\right)^k \left(\frac{4}{5}\right)^{\omega(d)}.$$

As $\omega(1) = 0$, it follows that

$$\int_{\mathbb{R}} \left| \sum_{\substack{n \in F \\ d|n}} \frac{1}{2^{\omega(n)} \sqrt{n}} \frac{1}{n^{it/2}} \right|^2 \frac{dt}{\pi(t^2 + 1)} \leq C \left(\frac{5}{4}\right)^k \frac{1}{2^{2\omega(d)} d} \left(\frac{4}{5}\right)^{\omega(d)} = C \left(\frac{5}{4}\right)^k \frac{1}{5^{\omega(d)} d}.$$

By reporting,

$$Y \leq \left(\frac{5}{2}\right)^k + C \left(\frac{5}{2}\right)^k \sum_{\substack{d \in F \\ d \neq 1}} \frac{1}{5^{\omega(d)} d}.$$

Now

$$\begin{aligned} \sum_{\substack{(n, m)=1 \\ n, m \in F}} \frac{1}{2^{\omega(m)+\omega(n)}} &= \sum_{m \in F} \frac{1}{2^{\omega(m)}} \sum_{\substack{n \in F \\ (n, m)=1}} \frac{1}{2^{\omega(n)}} = \sum_{m \in F} \frac{1}{2^{\omega(m)}} \left(\frac{3}{2}\right)^{k-\omega(m)} \\ &= \left(\frac{3}{2}\right)^k \sum_{m \in F} \frac{1}{3^{\omega(m)}} = \left(\frac{3}{2}\right)^k \left(\frac{4}{3}\right)^k = 2^k. \end{aligned}$$

Therefore,

$$N(B, F) \leq BY + \sum_{\substack{(n, m)=1 \\ n, m \in F}} 2^{k-\omega(m)-\omega(n)} \leq B \left(\frac{5}{2}\right)^k \left(1 + C \sum_{\substack{d \in F \\ d \neq 1}} \frac{1}{5^{\omega(d)} d}\right) + 4^k.$$

Now note that for $X \neq 0$,

$$(5.4) \quad \sum_{d \in F} \frac{1}{d} X^{-\omega(d)} = \prod_{j=1}^k \left(1 + \frac{1}{X p_j}\right).$$

By Weierstrass' inequality, if $0 < a_k < 1$ and $\sum_{k=1}^n a_k < 1$, then

$$1 + \sum_{k=1}^n a_k < \prod_{k=1}^n (1 + a_k) < \frac{1}{1 - \sum_{k=1}^n a_k}.$$

See Mitrinović [12, 3.2.37(3)]. Let $X > 0$ and assume that $\sum_{j=1}^k \frac{1}{p_j} \leq X/2$. Then

$$\prod_{j=1}^k \left(1 + \frac{1}{X p_j}\right) - 1 \leq \frac{\sum_{j=1}^k \frac{1}{X p_j}}{1 - \sum_{j=1}^k \frac{1}{X p_j}} \leq \frac{2}{X} \sum_{j=1}^k \frac{1}{p_j}.$$

We deduce that if $\sum_{j=1}^k \frac{1}{p_j} \leq \frac{5}{2}$, then

$$Y \leq \left(\frac{5}{2}\right)^k + C \left(\frac{5}{2}\right)^{k-1} \left(\sum_{j=1}^k \frac{1}{p_j}\right).$$

Consequently,

$$N(B, F) \leq B \left(\frac{5}{2} \right)^k \left\{ 1 + \frac{2C}{5} \left(\sum_{j=1}^k \frac{1}{p_j} \right) \right\} + 4^k \ll B \left(\frac{5}{2} \right)^k \left(1 + \sum_{j=1}^k \frac{1}{p_j} \right) + 4^k.$$

As $B \geq p_1 p_2 \dots p_k$, we have $(5/2)^k \ll B$ for large k . So that

$$N(B, F) \ll B \left(\frac{5}{2} \right)^k \left(1 + \sum_{j=1}^k \frac{1}{p_j} \right)$$

for large k , assuming $\sum_{j=1}^k \frac{1}{p_j} \leq \frac{5}{2}$.

Remark 5.1. Suppose that $\sum_{j=1}^k \frac{1}{p_j} \leq 1/2$. By Lemma 2.1,

$$N(B, F) \leq B^2 + 2B^2 \sum_{\substack{m \in F \\ m \neq 1}} \frac{1}{m} = B^2 \left(1 + 2 \frac{\sum_{j=1}^k \frac{1}{p_j}}{1 - \sum_{j=1}^k \frac{1}{p_j}} \right) \leq B^2 \left(1 + 4 \sum_{j=1}^k \frac{1}{p_j} \right).$$

We observe that the upper bound for $N(B, F)$ on special set is much better than the general case since

$$B \left(\frac{5}{2} \right)^k \left(1 + \sum_{j=1}^k \frac{1}{p_j} \right) \ll B^2 \left(1 + 4 \sum_{j=1}^k \frac{1}{p_j} \right).$$

6. Complements.

6.1. Multiplicative means of the divisor function. To bound from above $N_2(B)$, we directly estimated the following multiplicative sums appearing in (2.6)

$$(6.1) \quad \sum_{j \leq x} d(jn)d(jm),$$

in which $(n, m) = 1$. These sums are also interesting on their own for arbitrary positive integers n and m . We don't know any corresponding precise asymptotic estimate. The same question can be raised for the other sums $\sum_{j \leq x} d_k(jn)d_k(jm)$ and naturally for $\sum_{j \leq x} d^2(jm)$, $\sum_{j \leq x} d_k^2(jm)$.

Concerning however the "simple" sum $\sum_{j \leq x} d(jm)$, we indicate that

$$(6.2) \quad \sum_{n \leq x} d(nv) = \alpha(v)x(\log x + 2\gamma - 1) + \beta(v)x + \mathcal{O}(x^{1/3} \log x), \quad x \rightarrow \infty,$$

where v is any positive integer and

$$(6.3) \quad \alpha(v) = \sum_{\delta|v} \frac{\mu(\delta)}{\delta} d\left(\frac{v}{\delta}\right), \quad \beta(v) = - \sum_{\delta|v} \frac{\mu(\delta)}{\delta} (\log v) d\left(\frac{v}{\delta}\right).$$

See Wilson [20, Section 4]. The basic ingredient is nice formula (F) of Ramanujan [14],

$$d(uv) = \sum_{\delta|u, \delta|v} \mu(\delta) d\left(\frac{u}{\delta}\right) d\left(\frac{v}{\delta}\right).$$

See [14] and [20, (4.12), (4.15)]. The sum $\sum_{n \leq x} d^2(nv)$ can be estimated in the same manner as for getting (6.2) and essentially produces a main term of order $\alpha(v)x \log^3 x$. As by Busche-Ramanujan formula, the sums (6.1) directly rely upon the summatory function of $d(an^2)$, see (1.6), we now indicate some result concerning this one.

6.2. *Summatory function of $d(an^2)$.* Let $a \geq 1$ (the case $a = 1$ is known). One has the following formula.

Proposition 6.1. [16] *We have*

$$\sum_{n \leq x} d(an^2) = 2 \sum_{d \leq x} \varrho(d) \left[\frac{x}{d} \right] + \mathcal{O}(x),$$

where for all $d \geq 2$,

$$\varrho(d) = \prod_{p: v_p(d) \geq 1} p^{\left[\frac{v_p(d) + v_p(a) \wedge v_p(d)}{2} \right]}.$$

The application to sums (6.1) is studied in [16].

6.3. *$N(B, F)$.* The study of the restricted equation (1.8) for general FC sets F with $F \subset [1, B]$ is a quite interesting question, although we don't know much about the size of $N(B, F)$. We observe that the influence of F in the bound given in Lemma 2.1 is relatively weak. We recall that we have,

$$N(B, F) \leq 2B^2 \sum_{\substack{n \in F \\ n \neq 1}} \frac{1}{n} + B^2.$$

We indicate here a bound for $N(B, F)$ with factor B only and another factor depending on $\#(F)$ only. Let $F_m = \{d \in F : md \in F\}$, $m \in F$. Let also $\vartheta(m)$ denote the number of squarefree divisors of m . Then,

$$(6.4) \quad N(B, F) \leq (8\pi) B \sum_{m \in F} \vartheta(m) \#(F_m) + \#(F) \sum_{m \in F} \#(F_m).$$

We omit details of proof. It can be however verified that (6.4) yields weaker estimates if $F = [1, B]$ or in the case considered in Theorem 1.4, whose proof is based on an exact value of $N(B, F)$, see first equation of (5.1).

6.4. *Zeta sums.* Estimates for $N_k(B)$ allow one to provide lower bounds to local extrema of Zeta sums. An easy consequence can be stated as follows: There exist positive constants c_0, B_0, C_0 such that for any interval I and $B \geq B_0$,

$$(6.5) \quad \sup_{t \in I} \left| \sum_{n=1}^B n^{it} \right| \geq c_0 \sqrt{B \log B},$$

whenever $|I| \geq C_0 B^6 / \log B$. Similar bounds can be obtained for other Dirichlet sums related to $N_k(F, B)$ estimated in this paper. Note that the lower bound is trivial if $0 \in I$.

Hint: Uses the inequality

$$\frac{1}{|I|} \int_I \left| \sum_{n=1}^B n^{it} \right|^4 dt \leq \sup_{t \in I} \left| \sum_{n=1}^B n^{it} \right|^2 \left(\frac{1}{|I|} \int_I \left| \sum_{n=1}^B n^{it} \right|^2 dt \right).$$

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